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Bifurcations of piecewise smooth flows: Perspectives, methodologies and open problems

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ABSTRACT

In this paper, the theory of bifurcations in piecewise smooth flows is critically surveyed. The focus is on results that hold in arbitrarily (but finitely) many dimensions, highlighting significant areas where a detailed understanding is presently lacking. The clearest results to date concern equilibria undergoing bifurcations at switching boundaries, and limit cycles undergoing grazing and sliding bifurcations. After discussing fundamental concepts, such as topological equivalence of two piecewise smooth systems, discontinuity-induced bifurcations are defined for equilibria and limit cycles. Conditions for equilibria to exist in n-dimensions are given, followed by the conditions under which they generically undergo codimension-one bifurcations. The extent of knowledge of their unfoldings is also summarized. Codimension-one bifurcations of limit cycles and boundary-intersection crossing are described together with techniques for their classification. Codimension-two bifurcations are discussed with suggestions for further study.

1. Introduction

The theory of dynamical systems described by smooth ordinary differential equations is well developed (see e.g. [1]) but, for many systems of practical importance, the defining equations contain discontinuities. In such cases, the theoretical underpinning of some key results is uncertain.

In the gamut of dynamical systems with discontinuities, we must be careful to fix the class of systems of interest. The most general are hybrid systems, which are compounds of continuous and discrete dynamics (e.g. differential equations and maps); see for example [2–7]. Hybrid systems are too broad in scope to possess a substantially general bifurcation theory as yet. An important subclass of these are impact systems, where smooth evolution by a differential equation can be interrupted by a map from a discontinuity boundary to itself, such as the law of restitution for mechanical impact [8–15]. Grazing solutions, where the impact velocity is zero, provide insight into the dynamics near impact bifurcations. Such *grazing bifurcations* can be studied by analysing the local geometry of the impact manifold, using singularity theory [16] and the so-called discontinuity maps [17].

We restrict attention in this paper to a third important class, that of piecewise smooth flows, consisting of differential equations that are piecewise smooth and have what Filippov calls "discontinuous righthand sides" [18]. Discontinuities are isolated to a hypersurface, and unlike hybrid or impact systems, solutions are generally continuous, and moreover, are smooth everywhere except on the hypersurface. Our main interest is to describe the dynamical changes that result from invariant sets contacting such a discontinuity hypersurface. We assume no restriction on the degree of the discontinuity. If the system vector field jumps across the hypersurface then solutions may be non-differentiable there, either crossing the surface, or sticking to and sliding along it [18]. The latter is particularly challenging from a theoretical point of view, because solutions with segments of sliding can be non-unique or non-invertible.

Example. Consider an object moving on a surface with displacement x and velocity u, subject to an elastic force $-kx$ and Coulomb friction $-\mu\ \mathrm{sign}(u)$, where μ is the coefficient of friction [19]. Then x satisfies the piecewise smooth ordinary differential equations

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= -kx - \mu\ \mathrm{sign}(u), \\ \dot{t} &= 1. \end{aligned} \tag{1}$$

During *slipping*, the friction force has fixed magnitude μ, and points in the opposite direction to the velocity, switching at $u = 0$ to

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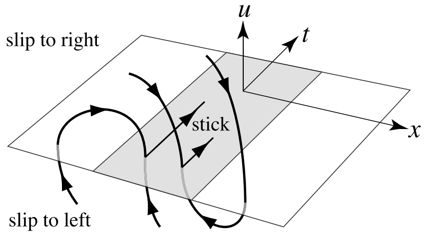

Fig. 1. The dry friction system (1). The friction force switches direction as u changes sign. In phase space, orbits slide along the surface $u = 0$ (shaded) where mechanical sticking occurs. Note that 'sliding' here corresponds to mechanical 'sticking', where $u = 0$, as opposed to 'slipping', where $u \neq 0$.

give the phase portrait in Fig. 1. If the speed reaches $u = 0$ at a time when $|kx| > \mu$, then the object crosses from leftward slip to rightward slip or vice versa, but, if $|kx| < \mu$, the object sticks to the surface. When this happens, solutions are said to "slide" along $u = 0$ in the t-direction of the phase space (x, u, t), meaning that the object sticks to the surface $u = 0$.

In this paper we survey the theory of *bifurcations* in piecewise smooth flows from a critical perspective. Rather than providing a comprehensive literature review, the aim is to summarize the extent of current knowledge, gathering together the more general results, and highlighting major areas requiring further work.

Piecewise smooth systems have been used for many years by engineers and physicists, long before being comprehensively formalized in mathematical terms. Perhaps as a consequence, knowledge of bifurcations in piecewise smooth systems is largely limited to specific examples, and does not yet approach the generality of bifurcation theory in smooth systems. A major obstacle to the development of a general theory is the inability to extend results in two or three dimensional systems to n dimensions, due to a lack of dimension reduction techniques, such as centre manifolds or normal form theory. Nevertheless, by drawing together results from the last half century within the framework provided by the differential inclusions of Filippov [18], the sewing (or "C") bifurcations of Feigin [20], and the generic singularities of Teixeira [21–23], we can begin the task of developing a coherent picture of the theory of bifurcations in piecewise smooth systems.

Piecewise smooth systems are now commonplace in models of real world dynamics. They are frequently treated by ad hoc modifications of tools borrowed from smooth systems; reviews can be found in [17,24–26]. A few of the wide range of applications that exhibit piecewise smooth dynamics include mechanical problems of friction [27,19,28–31], switched feedback in control theory [32–35] and electronics [36–40], nonsmooth models in economics [41,42], ecology [43–45], neuron signalling [46–48], genetic potentials [49–52], and novel nonlinear effects of superconductors [53,54]. Interest in such diverse applications amidst insufficiently developed theory has left behind a nomenclature that is dogged by semantic difficulties. This problem is bound to ease as theoretical advances take hold, and rigorous definitions begin to eliminate inconsistent uses of terminology. We shall pick our way through the more concrete definitions and most useful terminologies, giving reference to the alternatives only where it is useful to the discussion.

The layout of the paper is as follows. We first set out the fundamentals: the definition of a piecewise smooth system and its dynamics in Section 2, the forms of switching and sliding boundaries in Section 3, and topological equivalence between systems in Section 4. Then we introduce a geometrical treatment of more general discontinuity-induced bifurcations in Section 5. Existing results about the unfoldings of these bifurcations are reviewed in Section 6, and we discuss where new results are needed. Many results have been found for planar systems that await generalization to n dimensions; we review these in Section 7. By way of concluding remarks, we discuss some broader open problems and peer into a possible future of piecewise smooth dynamical systems in Section 8.

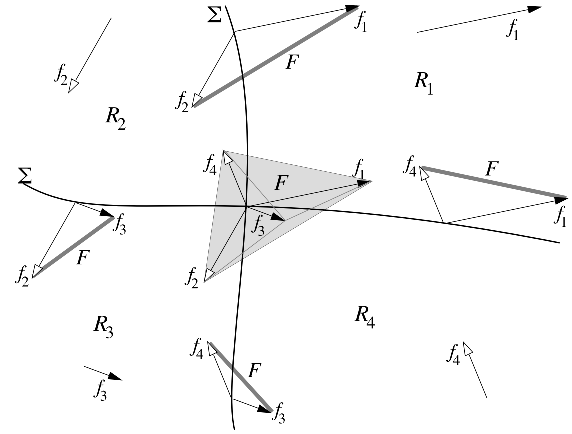

Fig. 2. A planar piecewise smooth system with four regions R_i, $i = 1, 2, 3, 4$, separated by a switching boundary Σ, where the vector field jumps between the values f_i. The righthand side of the differential inclusion, F (grey line/area), is set-valued on Σ, and its dimension depends on how many regions Σ is separating at each point (a convex hull of 2 vectors, except at the intersection where it is a convex hull of 4 vectors). Vectors f_1 and f_3 are shown with black arrowheads, f_2 and f_4 are shown with white arrowheads.

2. Dynamics of piecewise smooth systems

Definition 2.1. A piecewise smooth system consists of a finite set of ordinary differential equations

$$\dot{\mathbf{x}} = f_i(\mathbf{x}), \quad \mathbf{x} \in R_i \subset \mathbb{R}^n, \tag{2}$$

where the vector fields f_i are smooth, defined on disjoint open regions R_i, and are smoothly extendable to the closure of R_i. Regions R_i are separated by an $(n-1)$-dimensional set Σ called the *switching boundary*, which consists of finitely many smooth manifolds intersecting transversely. The union of Σ and all R_i covers the whole state space $D \subseteq \mathbb{R}^n$.

Away from Σ, the existence and uniqueness theorems of Picard and Lindelöf [55] ensure that solutions of (2) are well defined provided each f_i is sufficiently regular, but do not apply where the vector field is discontinuous, namely on Σ. Following Filippov [18], we overcome this problem by replacing (2) with a differential inclusion,

$$\dot{\mathbf{x}} \in F(\mathbf{x}) \tag{3}$$

where $F = f_i$ if $\mathbf{x} \in R_i$, and F is set-valued if $\mathbf{x} \in \Sigma$, given by the convex hull of $f_1, \ldots, f_m$ when $\mathbf{x}$ lies on the boundary of regions $R_1, \ldots, R_m$. A two dimensional inclusion with four regions is illustrated in Fig. 2.

Then we can define solutions to (2) as follows:

Definition 2.2. An absolutely continuous function $\mathbf{x}(t)$, with t in an open interval I, is a solution of (2) if and only if it satisfies the differential inclusion (3) almost everywhere in I.

This definition is found in [18], along with the proof that, under certain conditions, (at least) one solution passes through any point $\mathbf{x}$ of system (2). A more general discussion of differential inclusions is found in [56]. Notice that solutions need not be unique, as shown by the following example.

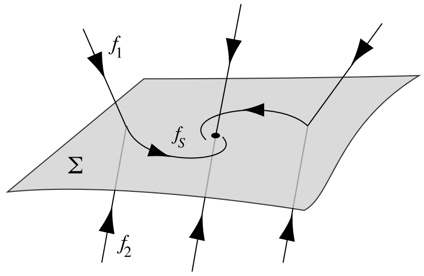

Fig. 3. A pseudoequilibrium in a two-region system occurs where the two vector fields point in opposite directions.

Example. Consider the one-dimensional system $\dot{x} = \operatorname{sign}(x)$, where Σ is the point $x = 0$. The righthand side of the corresponding differential inclusion is $F = 1$ where $x > 0$, $F = -1$ where $x < 0$, and $F = [-1, 1]$ at $x = 0$. Hence, at $x = 0$ the three solutions $x(t) = 0$, $x(t) = t$, and $x(t) = -t$ are admissible, as well as any solution that remains in 0 for a finite time and then departs, left or right, with unit speed.

As we see in the one-dimensional example above, a system may admit constant solutions on a switching boundary. In general, constant solutions of (2) come in two forms:

Definition 2.3. An *equilibrium* is a point where $f_i(\mathbf{x}) = 0$ for some i. A *pseudoequilibrium* is a point where $0 \in F(\mathbf{x})$, $\mathbf{x} \in \Sigma$.

Example. In n-dimensions, let the vector field change between f_1 (in R_1) and f_2 (in R_2) across Σ. Then the differential inclusion (3) becomes

$$\dot{\mathbf{x}} \in F = \{\lambda f_1 + (1-\lambda) f_2\}, \tag{4}$$

where $\lambda = 1$ in R_1, $\lambda = 0$ in R_2, and $\lambda \in [0, 1]$ on Σ. By (4), pseudoequilibria appear when f_1 and f_2 are linearly dependent and point in opposite directions, as illustrated in Fig. 3.

Over the last thirty years, piecewise smooth systems have been redefined a number of times in slightly different ways. Definitions 2.1 and 2.2 are the simplest and most commonly used among those considered in [18]. They are similar to another definition, albeit restricted to two-dimensional systems, given in [57]. The term pseudoequilibrium in Definition 2.3, introduced in [58], is now quite standard in the literature.

3. Boundaries

To study the dynamical features that distinguish smooth and piecewise smooth dynamical systems, we concentrate on the geometry of solutions at or near the switching boundary Σ. To this end, it is convenient to introduce three mathematical tools: a function to describe Σ, a derivative to detect tangencies between solutions and boundaries, and an explicit formula for the component of F along Σ in the cases when it exists and is unique.

We represent Σ as the zero set of a scalar function $h : \mathbb{R}^n \mapsto \mathbb{R}$, with

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) = 0\}. \tag{5}$$

At points where Σ is a smooth hypersurface, we assume that h is smooth and has nonzero gradient $h_{,\mathbf{x}}$ (subscripts will denote a derivative only when preceded by a comma). Notice that the global smoothness and differentiability of h is not a concern, since its gradient will only be needed in local analysis.

We then write the directional derivative of h with respect to the vector field in terms of Lie derivatives $\mathcal{L}_{f_i} h = h_{,\mathbf{x}} \cdot f_i$. The m-th order Lie derivative will be written as $\mathcal{L}^m_{f_i} h$, e.g., $\mathcal{L}^2_{f_i} h = \mathcal{L}_{f_i}(\mathcal{L}_{f_i} h)$.

Definition 3.1. A *sliding vector* is any vector $f_s(\mathbf{x}) \in F$ that lies tangent to Σ for $\mathbf{x} \in \Sigma$.

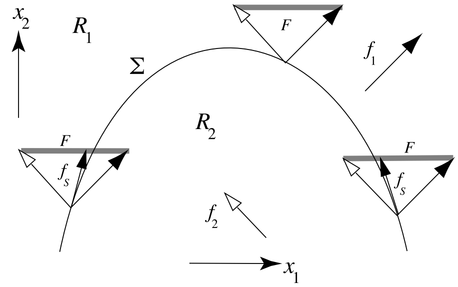

Fig. 4. A planar piecewise smooth vector field that switches between f_1 in region R_1 and f_2 in region R_2. At the switching boundary Σ we consider the inclusion F. This gives sliding/escaping if F contains an element f_s tangent to Σ, and crossing otherwise.

According to Definition 2.2, solutions of system (2) that reach Σ may *cross* through Σ if F contains no sliding vectors, or *slide* along Σ if F contains a sliding vector. Thus the switching boundary is partitioned into three different regions as follows.

Definition 3.2.

- In a *crossing region*, F contains no sliding vectors.
- In a *sliding region*, F everywhere contains at least one sliding vector, and all neighbouring vector fields f_i point towards Σ.
- In an *escaping region*, F everywhere contains at least one sliding vector, and at least one of the neighbouring vector fields f_i point into its corresponding region R_i.

As an example of the definition for an escaping region, consider Fig. 2: the boundary separating regions R_1 and R_2 is an escaping region where f_1 and f_2 both point away from Σ, but the boundary intersection is also an escaping region where f_1 points away from Σ but all others points towards it.

The distinction between sliding and escaping regions is important: at a sliding region all solutions are confined to Σ in forward time, while at an escaping region solutions may either continue sliding or be ejected from Σ. Because of this dual nature of sliding and escaping, they are sometimes referred to respectively as stable and unstable sliding (see for example [59]).

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= \operatorname{sign}(x_2 + x_1^2), \\ \dot{x}_2 &= 1, \end{aligned} \tag{6}$$

sketched in Fig. 4. The switching boundary Σ is the curve $x_2 = -x_1^2$, and we let R_1 be the region above Σ, with R_2 below. The righthand side of the differential inclusion, F, is sketched in Fig. 4 with the sliding vectors f_s, and the corresponding dynamics is shown in Fig. 5. The boundaries between crossing and sliding/escaping occur at the tangencies T_1 and T_2, where the tangent vector to Σ, given by $(1, -2x_1)$, lies along f_1 and f_2 respectively. Then escaping takes place on Σ to the right of T_2, and sliding to the left of T_1.

In general, boundaries between crossing, sliding, and escaping regions can occur either where Σ is nonsmooth, which we call *boundary intersections*, or where Σ is smooth but tangent to one of the f_i, satisfying the *tangency conditions*

$$\mathcal{L}_{f_i} h = 0. \tag{7}$$

Away from boundary intersections, we can write the vector field near a switching boundary $h = 0$ as

$$\dot{\mathbf{x}} = \begin{cases} f_1(\mathbf{x}) & \text{if } h(\mathbf{x}) > 0, \\ f_2(\mathbf{x}) & \text{if } h(\mathbf{x}) < 0. \end{cases} \tag{8}$$

The differential inclusion for (8) is then given by (4). A sliding vector, from Definition 3.1, is the element of (4) tangent to Σ, which fixes $\lambda = \mathcal{L}_{f_1} h(\mathbf{x}) / (\mathcal{L}_{f_2} h(\mathbf{x}) - \mathcal{L}_{f_1} h(\mathbf{x}))$, giving the sliding vector field

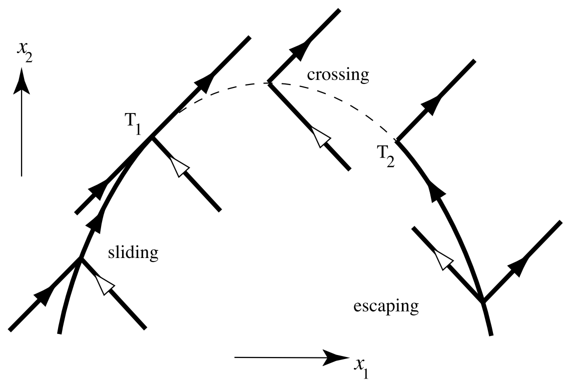

Fig. 5. The piecewise smooth dynamics corresponding to Fig. 4. The crossing region is dashed. A sliding segment sticks to Σ in the sliding and escaping regions, which are bounded by the tangency points T_1, T_2.

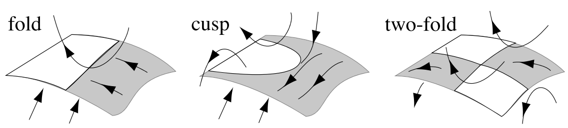

Fig. 6. Basic tangencies between a piecewise smooth vector field and a switching boundary: the fold, where the vector field has a quadratic tangency to Σ; the cusp, where the vector field has a cubic tangency to Σ, and the sliding vector field has a quadratic tangency to the sliding (or escaping) boundary; the two-fold, where the vector field has a quadratic tangency to Σ on both sides. Sliding/escaping regions are shaded, crossing regions are unshaded.

$$f_s(\mathbf{x}) = \frac{\mathcal{L}_{f_2}h(\mathbf{x})f_1(\mathbf{x}) - \mathcal{L}_{f_1}h(\mathbf{x})f_2(\mathbf{x})}{\mathcal{L}_{f_2}h(\mathbf{x}) - \mathcal{L}_{f_1}h(\mathbf{x})}, \quad \mathbf{x} \in \Sigma. \tag{9}$$

Sliding or escaping occur when $\lambda \in [0, 1]$, and hence the dynamics in a sliding/escaping region is given by

$$\dot{\mathbf{x}} = f_s(\mathbf{x}). \tag{10}$$

As emphasized by Filippov [18] and Teixeira [23], tangencies are central to understanding dynamics at a switching boundary and, as we have seen, they form the boundaries between regions of crossing, sliding, and escaping, on Σ. The three simplest types of tangency that are encountered on smooth portions of Σ dividing regions R_1 and R_2, are (see Fig. 6):

- the *fold* (quadratic tangency), where $\mathcal{L}_{f_1}h = 0$, while $\mathcal{L}^2_{f_1}h \neq 0$, $\mathcal{L}_{f_2}h \neq 0$, and the gradient vectors of h and $\mathcal{L}_{f_1}h$ are linearly independent.
- the *cusp* (cubic tangency), where $\mathcal{L}_{f_1}h = \mathcal{L}^2_{f_1}h = 0$, while $\mathcal{L}^3_{f_1}h \neq 0$, $\mathcal{L}_{f_2}h \neq 0$, and the gradient vectors of h, $\mathcal{L}_{f_1}h$ and $\mathcal{L}^2_{f_1}h$ are linearly independent.
- the *two-fold* (double tangency), where $\mathcal{L}_{f_1}h = \mathcal{L}_{f_2}h = 0$, while $\mathcal{L}^2_{f_1}h \neq 0$, $\mathcal{L}^2_{f_2}h \neq 0$, and the gradient vectors of h, $\mathcal{L}_{f_1}h$ and $\mathcal{L}_{f_2}h$ are linearly independent.

The simplest tangency is the fold. Given a switching boundary $x_1 = 0$ (so $h = x_1$) in coordinates $\mathbf{x} = (x_1, x_2)$ for a planar system, a fold is defined [18] as the set $x_1 = x_2 = 0$ in the local normal form

$$\begin{aligned} f_1 &= \pm(s_1x_2, 1), \\ f_2 &= \pm(1, 0), \end{aligned} \tag{11}$$

where s_1 takes values ± 1. This can be easily extended to n dimensions by demanding that a fold has the normal form

$$\begin{aligned} f_1 &= \pm(s_1x_2, 1, 0, \ldots), \\ f_2 &= \pm(1, 0, 0, \ldots), \end{aligned} \tag{12}$$

where the dots denote a sequence of zeros. If s_1 is positive [or negative] then the flow in R_1 curves away from [towards] Σ, which

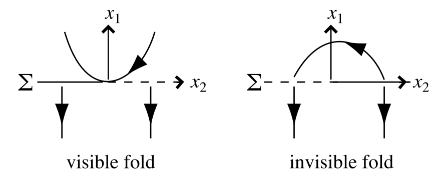

Fig. 7. The fold as a boundary between escaping (full line) and crossing (dashed line), illustrated in two dimensions. It can be either visible (left) or invisible (right). By reversing arrows we would swap escaping for sliding.

we call a *visible* [*invisible*] fold. The region $x_2 < 0$ on Σ is a sliding region if we take the '+' signs in (12), and an escaping region if we take the '−' signs; the latter is illustrated in Fig. 7. The case of a visible fold at the boundary of a sliding region in three dimensions is shown in Fig. 6.

Consider now the remaining tangencies listed in Fig. 6, namely the cusp or the two-fold. Given again a switching boundary $x_1 = 0$ and with dots denoting sequence of zeros, the cusp can be defined, following [23], as the set $x_1 = x_2 = x_3 = 0$ in the local normal form

$$\begin{aligned} f_1 &= \pm(x_3 + s_1x_2^2, 1, 0, \ldots), \\ f_2 &= \pm(1, 0, 0, \ldots), \end{aligned} \tag{13}$$

where s_1 takes values ± 1. There are branches of visible and invisible folds along the $x_2 > 0$ and $x_2 < 0$ branches of $x_1 = x_3 + s_1x_2^2 = 0$. Similarly to the fold, we can classify cusps either as invisible or visible (following [59]), according to whether the sliding flow curves towards ($s_1 = +1$) or away from ($s_1 = -1$) the sliding boundary respectively. The cusp appears generically in systems of three or more dimensions. The case of a visible case in three dimensions is shown in Fig. 6.

The two-fold can be defined as the set $x_1 = x_2 = x_3 = 0$ on a switching boundary $x_1 = 0$, in the local normal form

$$\begin{aligned} f_1 &= (s_1x_2, 1, a, \ldots), \\ f_2 &= (s_2x_3, b, 1, \ldots), \end{aligned} \tag{14}$$

where the dots denote sequence of zeros, a and b are constants such that the sliding vector field is structurally stable (requiring $ab \neq 1$), and the s_i are signs ± 1 determining whether the folds are visible or invisible. This extends the three-dimensional normal forms given in [18,23] to arbitrary dimensions. The fold along $x_2 = 0$ is visible [resp. invisible] if s_1 is positive [negative], and the fold along $x_3 = 0$ is visible [invisible] if s_2 is negative [positive]. The two-fold appears generically in systems of three or more dimensions. The visible–visible case (two visible folds) in three dimensions is depicted in Fig. 6.

A general treatment of sliding boundary topology in $n > 3$ dimensions has not yet been carried out, but a basic understanding of the fold, cusp, and two-fold is sufficient for many scenarios that arise in the literature on piecewise smooth systems, and they will play a major role in the remainder of this paper. We summarize their coordinate independent defining conditions in Table 1.

In this section we discussed how tangencies form the boundaries of crossing/sliding/escaping, but they have a further role, as points through which special solutions (equilibria, limit cycles, etc.) can alter the topology of their intersection with the switching boundary. Before we can discuss bifurcations in piecewise smooth systems further, we must define the conditions under which two piecewise smooth systems are topologically equivalent.

4. Equivalence of piecewise smooth systems

To discuss the topological properties of piecewise smooth systems we need to distinguish between different types of solution. Following a definition made in [57], we call a *segment* any smooth

Table 1
Defining conditions of generic tangencies in $n \geq 3$, given the system (8). In addition, non-degeneracy requires that the gradients of the following quantities with respect to $\mathbf{x}$ are linearly independent: h and $\mathcal{L}_{f_1}h$ for the fold; h, $\mathcal{L}_{f_1}h$ and $\mathcal{L}^2_{f_1}h$ for the cusp; h, $\mathcal{L}_{f_1}h$ and $\mathcal{L}_{f_2}h$ for the two-fold. The derivatives in column three must also be non-zero.

Tangency	Definition	Sliding/escaping, and visibility type
Fold	$\mathcal{L}_{f_1}h = 0$	Visible if $\mathcal{L}^2_{f_1}h > 0$, invisible if $\mathcal{L}^2_{f_1}h < 0$ Sliding if $\mathcal{L}_{f_2}h > 0$, escaping if $\mathcal{L}_{f_2}h < 0$
Cusp	$\mathcal{L}_{f_1}h = 0$ and $\mathcal{L}^2_{f_1}h = 0$	Visible if $(\mathcal{L}^3_{f_1}h)(\mathcal{L}_{f_2}h) < 0$, Invisible if $(\mathcal{L}^3_{f_1}h)(\mathcal{L}_{f_2}h) > 0$ Sliding if $\mathcal{L}_{f_2}h > 0$, escaping if $\mathcal{L}_{f_2}h < 0$
Two-fold	$\mathcal{L}_{f_1}h = 0$ and $\mathcal{L}_{f_2}h = 0$	Visible-visible if $\mathcal{L}^2_{f_1}h > 0 > \mathcal{L}^2_{f_2}h$ Invisible-invisible if $\mathcal{L}^2_{f_1}h < 0 < \mathcal{L}^2_{f_2}h$ Visible-invisible if $(\mathcal{L}^2_{f_1}h)(\mathcal{L}^2_{f_2}h) > 0$

solution $\mathbf{x}(t)$ that is entirely contained in a region R_i or in a sliding or escaping region, and defined for any open time interval $t \in I$. We refer to maximal segments if I is maximal. In order to distinguish between segments that slide on the switching boundary and segments that lie in regions R_i, we call the former *sliding segments* and the latter *non-sliding segments*. Finally, an *orbit* is a continuous concatenation of closures of segments. We assume typically that an orbit is maximal.

Because segments are solutions of a smooth vector field, maximal segments do not overlap in the state space, and therefore an equivalence between systems can be defined segment-wise as a bijection between sets of disjoint elements. Thus, we define the concept of topological equivalence for piecewise smooth systems as follows.

Definition 4.1. Two piecewise smooth systems are *topologically equivalent* if there exists a homeomorphism sending maximal segments of one system onto maximal segments of the other, preserving the direction of time, and preserving whether a segment is sliding, escaping, crossing, or not in contact with Σ.

A consequence of this definition of topological equivalence is that switching boundaries are mapped to switching boundaries. The same is true of alternative definitions of topological equivalence, based on orbits rather than segments (and thus establishing relations between sets of non-disjoint elements), given in [18,60,17]. It has been pointed out in [61,62] that preserving Σ in this way makes a stronger restriction than is necessary to define a topological equivalence (hence those authors call these Σ-equivalences). A weaker topological equivalence can be defined which does not preserve the switching boundary, but does preserve sliding; let us call this 'sliding equivalence'. Two examples are illustrated in Fig. 8: the figures on each row are *not* topologically (Σ-) equivalent by Definition 4.1, but are sliding equivalent. While useful from a pure mathematical perspective, this sliding equivalence does not distinguish between systems with different crossing topologies. The sliding equivalence is therefore inappropriate for applications where crossing between different regions is of physical interest. Because the intersection of orbits with the switching boundary will be extremely important in later sections, we use exclusively Definition 4.1.

Using Definition 4.1, we define the notion of bifurcation in a piecewise smooth system as follows.

Definition 4.2. A bifurcation occurs if an arbitrarily small perturbation produces a topologically non-equivalent system. The bifurcation is *discontinuity-induced* if it affects the state portrait in more than one region, or in Σ.

A further distinction can be made if we describe discontinuity-induced bifurcations as *strong* if they involve non-generic configuration of orbits with respect to switching boundaries, or *weak* if

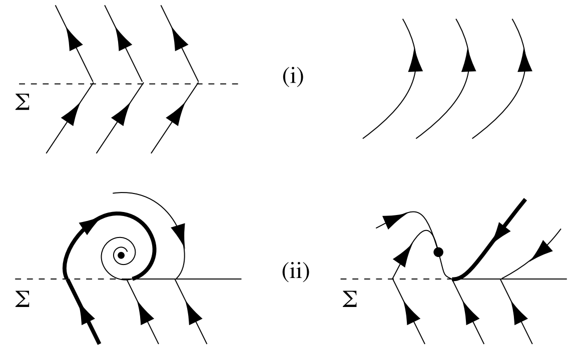

Fig. 8. Piecewise smooth topological equivalences. Each row (i) and (ii) shows two systems which are not topologically equivalent ('Σ equivalent') by Definition 4.1, which respects dynamics in both the sliding (full line) and crossing (dashed line) regions, but are 'sliding equivalent', which neglects the crossing region. In (i), orbits cross Σ in one system but not in the other, while in (ii), the orbit tangent to Σ, shown bold, crosses in one system but not in the other.

the presence of a switching boundary is incidental and the bifurcation can be treated using the mathematical tools of smooth maps and flows [6]. The former of these will be our exclusive concern, as they involve the switching boundary in a nontrivial way. Examples of the latter include a cycle undergoing a bifurcation at a switching boundary but expressible by a smooth Poincaré map throughout, or a pseudoequilibrium undergoing a bifurcation of saddle–node type of the smooth vector field f_s.

5. Geometric overview of discontinuity-induced bifurcations

We are now in a position to consider bifurcations involving local and global dynamics. With very few exceptions, the discontinuity-induced bifurcations that have been most analysed in the literature are those affecting equilibria, pseudoequilibria, and limit cycles, so we focus here on these. We will classify systems by studying generic configurations of orbits, and define as generic any configuration that satisfies a certain (finite) number of inequalities, referred to as *genericity* conditions. Thus, away from boundary intersections, considering a system defined as in (8), we have that generically:

G1. if there exists an equilibrium $\bar{\mathbf{x}}$, it lies in the interior of a region R_i, so that $h(\bar{\mathbf{x}}) \neq 0$,
G2. if there exists a pseudoequilibrium $\bar{\mathbf{x}}$, it lies in the interior of a sliding or escaping region, so that $h(\bar{\mathbf{x}}) = 0$ and $\mathcal{L}_{f_i}h(\bar{\mathbf{x}}) \neq 0$ for all i,
G3. if a non-sliding segment passing through a given point $\mathbf{x} \notin \Sigma$ reaches Σ at $\bar{\mathbf{x}}$, it does so in the interior of a sliding or crossing region, so $\mathcal{L}_{f_i}h(\bar{\mathbf{x}}) \neq 0$ for all i,
G4. if a sliding segment passing through a given point $\mathbf{x} \in \Sigma$ reaches a boundary of the sliding or escaping region at $\bar{\mathbf{x}}$, it does so at a fold (so $\mathcal{L}_{f_i}h(\bar{\mathbf{x}}) = 0$ and $\mathcal{L}^2_{f_i}h(\bar{\mathbf{x}}) \neq 0$ for some i, and $\mathcal{L}_{f_j}h(\bar{\mathbf{x}}) \neq 0$ for all other $j \neq i$).

These are illustrated in Fig. 9. A fifth case should be added here about which very little is known, namely a sliding segment that reaches a two-fold. This almost fits into G4 above, but violates the condition $\mathcal{L}_{f_j}h \neq 0$ for some j, and instead satisfies $\mathcal{L}_{f_j}h = 0$ with $\mathcal{L}^2_{f_j}h \neq 0$. In three dimensions (see Fig. 10) it has been shown that a sliding segment through a given point $\mathbf{x} \in \Sigma$ can generically hit a two-fold [63], a fact which is not immediately obvious, but follows because the possible topologies of the sliding vector field (9) include, as one case, the phase portrait shown in Fig. 10. Two-folds in higher dimensions have not been studied at all (see Section 8). To highlight this open problem we include the case that, generically:

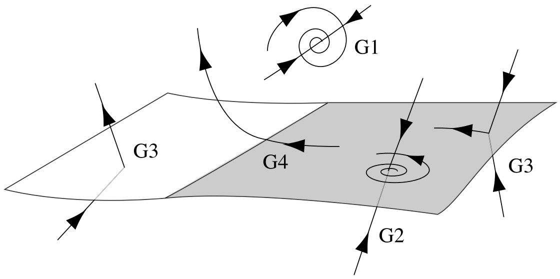

Fig. 9. Generic dynamics in a piecewise smooth system: G1 is an equilibrium away from Σ, G2 is a pseudoequilibrium, G3 is a non-sliding segment hitting Σ away from the boundary of sliding, G4 is a sliding segment hitting a fold transversely. The sliding region is shaded, the crossing region is unshaded.

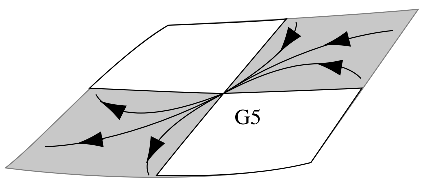

Fig. 10. Generic dynamics in a piecewise smooth system: G5 consists of an open region of sliding segments that hit a two-fold. Sliding/escaping regions are shaded, crossing regions are unshaded.

G5. if a sliding segment passing through a given point $\mathbf{x} \in \Sigma$ reaches a boundary of the sliding or escaping region in a scenario other than G4, it hits a two-fold.

G1–G5 do not provide a complete list of all the configurations of orbits that are generic in piecewise smooth systems, but describe geometric rules that generically must be adhered to near a switching boundary, by limit cycles and equilibria, in the absence of boundary intersections. By violating any one of the inequalities above we obtain a discontinuity-induced bifurcation. As stated in Section 4, (strong) discontinuity-induced bifurcations imply non-generic configurations of invariant sets and boundaries. Specifically, bifurcations of equilibria or pseudoequilibria occur when they collide with a switching or sliding boundary, violating G1–G2, while bifurcations of limit cycles occur when a cycle intersects a switching or sliding boundary non-generically, violating G3–G5. Since the nature of these bifurcations is essentially geometric, geometry can be used to catalogue them. In the following section, we firstly discuss codimension-one bifurcations, occurring when one of the genericity conditions is violated, and then move on to discuss a starting point for classifications of codimension-two bifurcations. In both cases, we address first bifurcations of equilibria and pseudoequilibria, then those of limit cycles.

5.1. Codimension-one discontinuity-induced bifurcations

5.1.1. Equilibria and Pseudoequilibria

Discontinuity-induced bifurcations of equilibria and pseudoequilibria occur whenever the genericity conditions G1 and G2 at the start of Section 5 are violated, implying that an equilibrium meets the switching boundary Σ, or a pseudoequilibrium meets the boundary of a sliding or escaping region. These can happen in two ways:

B1. the simultaneous collision of an equilibrium and a pseudoequilibrium at the boundary of a sliding/escaping region, called a boundary equilibrium,
B2. collision of a pseudoequilibrium with a two-fold.

It may not be immediately obvious that B1 and B2 should be the only generic ways that a pseudoequilibrium can hit the boundary of a sliding/escaping region, or that B1 should be the only generic way an equilibrium can hit Σ. These are facts that have not previously come to light in the literature. They are, however, an immediate consequence of the generic local geometry, as we now briefly explain.

Regarding B1, if an equilibrium, $\bar{\mathbf{x}}$, is a generic zero of f_1 (without loss of generality), then it lies at the intersection of a pair of curves given by the following two sets of equations:

$$\mathcal{L}_{f_1} h(\mathbf{x}) = \mathcal{L}_{f_1}^2 h(\mathbf{x}) = \cdots = \mathcal{L}_{f_1}^{n-1} h(\mathbf{x}) = 0, \tag{15}$$

where n is the dimension of the system, i.e. $\mathbf{x} \in \mathbb{R}^n$, and

$$f_1(\mathbf{x}) = \eta(\mathbf{x}) f_2(\mathbf{x}), \tag{16}$$

for some scalar function $\eta(\mathbf{x})$, with $\eta(\bar{\mathbf{x}}) = 0$. For $\bar{\mathbf{x}} \in \Sigma$, these imply that a boundary equilibrium coincides with both a pseudoequilibrium and an nth order tangency between f_1 and Σ. To see this, consider what happens if a system with $\bar{\mathbf{x}} \in \Sigma$ is perturbed. There generically exist two points nearby, say $\mathbf{x}_T \in \Sigma$ and $\mathbf{x}_P \in \Sigma$, that satisfy (15) and (16) respectively; $\mathbf{x}_T$ is an n-th order tangency between the flow of f_1 and Σ, and $\mathbf{x}_P$ is a pseudoequilibrium of f_s if $\eta < 0$ (if $\eta > 0$ it is a crossing point). The sign of η at $\mathbf{x}_P$ changes as the equilibrium passes through it, and therefore the inequality $\eta < 0$ is always satisfied – and a pseudoequilibrium exists – on one side of the bifurcation. The nth order tangency at $\mathbf{x}_T$, satisfying (15), persists on both sides of the bifurcation. Part of this analysis is contained, with further details, in [64].

Regarding B2, at a pseudoequilibrium (16) must be satisfied. If the pseudoequilibrium collides with the boundary of its sliding/escaping region, then either f_1 or f_2 vanishes, which is the boundary equilibrium bifurcation B1, or both f_1 and f_2 are tangent to the switching boundary, which occurs at folds. Generically, these intersect transversely and form a two-fold. This bifurcation has been studied for the first time in [65] in three-dimensional systems, but an n-dimensional analysis has not yet been carried out.

The conditions that must be satisfied for B1 and B2 to be generic have not been rigorously studied in n-dimensions, (with a few exceptions, see Section 5.2.1). However, the discussion above implies that B1 and B2 must satisfy the following genericity conditions:

B1. the equilibrium and pseudoequilibrium are hyperbolic, hit the boundary transversely and, letting the equilibrium belong to R_1 without loss of generality, then the equilibrium hits Σ where $\mathcal{L}_{f_2} h \neq 0$, and where the gradient vectors of h and each $\mathcal{L}_{f_1}^m h(\mathbf{x})$ for $m = 1, \ldots, n-1$, are linearly independent,
B2. the pseudoequilibrium is hyperbolic, crosses from sliding to escaping regions, and does so where $\mathcal{L}_{f_1}^2 h \neq 0$ and $\mathcal{L}_{f_2}^2 h \neq 0$.

We discuss the unfoldings of these in Section 6.1.

5.1.2. Limit cycles

Discontinuity-induced bifurcations of limit cycles occur when conditions G3 and G4 at the start of Section 5 are violated. This happens when:

B3. a non-sliding segment of a cycle reaches a boundary at a fold point.
B4. a sliding segment of a cycle reaches a boundary of the sliding/escaping region at a cusp.

These both involve a limit cycle grazing (being quadratically tangent to) a boundary: a non-sliding segment grazing Σ in B3, and a sliding segment grazing the sliding/escaping boundary in B4. The list is clearly incomplete. If we relax the genericity condition G4, either $\mathcal{L}_{f_1}^2 h \neq 0$ is violated in which case we can obtain B4 above, or $\mathcal{L}_{f_2} h \neq 0$ is violated in which case the cycle intersects a two-fold. At first this appears to be in contradiction to the

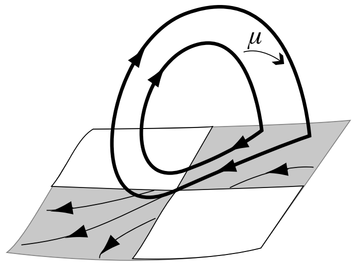

Fig. 11. Generic dynamics in a piecewise smooth system. Except in the case G5 (Fig. 10), a limit cycle will not generically hit a two-fold, but may do so as a parameter μ is varied.

case G5 in Section 5. However, it is known in three dimensions (see for example [18,23,59,65]) that, depending on the topology of the sliding vector field, such an intersection can occur, either generically (as in Fig. 10) or in the unfolding of a one-parameter bifurcation (as in another case of the two-fold, shown in Fig. 11). We therefore must add, as we did in Section 5, a speculative case of codimension-one discontinuity-induced bifurcation of limit cycles, that occurs when:

B5. a sliding segment of a cycle reaches a boundary of the sliding/escaping region at a two-fold.

The codimension-one bifurcation scenarios B3–B5 are known in the literature as *sliding bifurcations*. In [17,66] four types of sliding bifurcation are found, under the hypothesis that the bifurcating cycle has no sliding or escaping segments away from the tangency to Σ. In [59] the local geometry is analysed, revealing that four additional scenarios of sliding bifurcations involving escaping regions are possible. We introduce unfoldings for all of these in Section 6.2.

Similarly to B1–B2, the conditions that must be satisfied for B3–B5 to be generic have not been rigorously studied in n-dimensions. However, a brief inspection suggests that they should satisfy the following genericity conditions:

B3. the cycle is hyperbolic, and does not involve non-generic intersections outside the neighbourhood of the fold, where $\mathcal{L}_{f_1}^2 h \neq 0$, $\mathcal{L}_{f_2} h \neq 0$,
B4. the cycle is hyperbolic, and does not involve non-generic connections outside the neighbourhood of the cusp, where $\mathcal{L}_{f_1}^3 h \neq 0$, $\mathcal{L}_{f_2} h \neq 0$.

For case B5 there is insufficient theory to speculate about genericity conditions in n dimensions, other than that the cycle should not involve non-generic intersections outside the neighbourhood of the two-fold.

5.2. Codimension-two discontinuity-induced bifurcations

The genericity conditions in the previous section suggest that certain codimension-two bifurcation scenarios are also possible. which have come to light in specific circumstances.

5.2.1. Equilibria and pseudoequilibria

Codimension-two discontinuity-induced bifurcations of equilibria and pseudoequilibria occur whenever the genericity conditions for B1 and B2 in Section 5.1.1 are violated. Considering first B1, this can happen in four ways:

B1.1. the equilibrium or pseudoequilibrium is nonhyperbolic when it hits the boundary,
B1.2. the equilibrium or pseudoequilibrium grazes (is quadratically tangent to) the boundary,
B1.3. an equilibrium of f_1 hits the boundary where f_2 is tangent to Σ,
B1.4. the equilibrium collides with a point of tangency of order $n+1$.

Considering B2, the genericity conditions can be violated in three ways:

B2.1. the pseudoequilibrium is nonhyperbolic when it hits the two-fold,
B2.2. the pseudoequilibrium grazes the two-fold,
B2.3. a pseudoequilibrium collides with a cusp.

Unfoldings for these will be discussed in Section 6.3.

5.2.2. Limit cycles

The genericity conditions B3–B5 in Section 5.1.2 can be violated in three essentially different ways, which were classified in [67] as:

Type I: the inequalities on $\mathcal{L}_{f_i}^n h$ are violated,
Type II: the cycle is nonhyperbolic or has homoclinic/heteroclinic connections to equilibria/pseudoequilibria,
Type III: the cycle has more than one grazing with a switching boundary, or boundary of a sliding/escaping region.

Unfoldings for these will be discussed in Section 6.4.

5.3. Boundary-intersection crossing bifurcations

Dynamics of a piecewise smooth system at a boundary intersection can be rather complex. One reason for this is that the set F in (3) may contain an infinite number of vectors lying in the tangent space of the boundary intersection. Attempts to define simplified dynamics in such cases have been made in [68–70]. In the presence of a transverse intersection between finitely many switching boundaries, the genericity conditions listed at the beginning of Section 5 must be amended:

$\tilde{G}1$. if there exists an equilibrium $\bar{\mathbf{x}}$, it lies in the interior of a region R_i, so that $h(\bar{\mathbf{x}}) \neq 0$,
$\tilde{G}2$. if there exists a pseudoequilibrium $\tilde{\mathbf{x}}$, it lies either: in the interior of a sliding or escaping region, away from tangencies (e.g. folds and cusps), or at a boundary intersection.
$\tilde{G}3$. if a non-sliding segment passing through a given point $\mathbf{x} \notin \Sigma$ reaches Σ at $\tilde{\mathbf{x}}$, it does so in the interior of a sliding or crossing region, away from tangencies and boundary intersections,
$\tilde{G}4$. if a sliding segment passing through a given point $\mathbf{x} \in \Sigma$ reaches a boundary of the sliding or escaping region, it does so at a fold, a two-fold, or at an intersection between two smooth portions of Σ.

These suggest themselves as reasonable conditions for genericity in the presence of boundary intersections, but it has not yet been proven that they are sufficient. Furthermore, bifurcations obtained by violating conditions $\tilde{G}1$, $\tilde{G}2$, or $\tilde{G}4$, appear to have never been studied. All existing results concern the violation of conditions $\tilde{G}3$, when a non-sliding segment of a limit cycle reaches a boundary intersection. In this case, we must assume that the cycle is hyperbolic, involves only generic intersections outside the neighbourhood of the boundary intersection, that none of the neighbouring vector fields is tangent to any one of the two smooth portions of Σ, and that the intersection involves only two smooth portions of Σ.

As this stage, the available results on bifurcations involving boundary intersections are insufficient to provide a list of the possible codimension-two bifurcations.

6. Unfoldings

Following on from the methods used to define the bifurcation scenarios in Section 5, we begin to unfold them by analysing

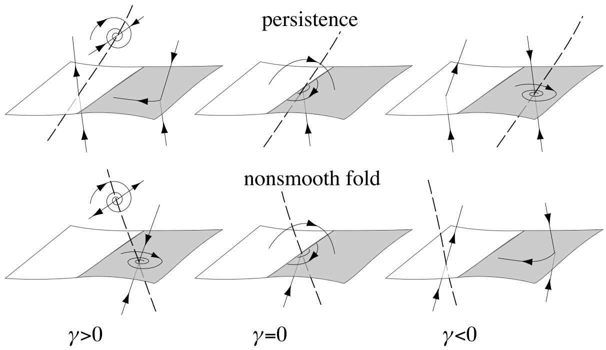

Fig. 12. The persistence and nonsmooth fold scenarios of boundary equilibrium bifurcations. In any number of dimensions, the functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are linearly dependent along a one dimensional curve (dashed) given by (16). If this intersects the sliding region (shaded) then a pseudoequilibrium exists, while an equilibrium exists if $\gamma > 0$.

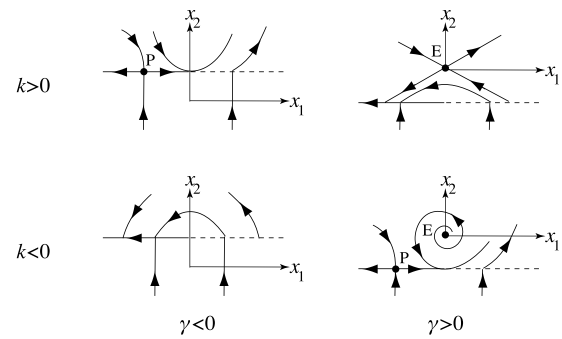

Fig. 13. Persistence ($k > 0$) and nonsmooth-fold ($k < 0$). Phase portraits of (20) with a switching boundary at $x_2 = -\gamma$, a sliding region over $x_1 < 0$ (full line) and crossing over $x_1 > 0$ (dashed line). Equilibria (E) lie at $(x_1, x_2) = (0, 0)$ and pseudoequilibria (P) at $(x_1, x_2) = \gamma(k, -1)$.

the local geometry. Most examples of unfoldings in the literature analyse specific systems in low dimensions, and few have been studied in a way that generalizes to n dimensions. In this section we focus only on the discussion of unfoldings that apply in n dimensional piecewise smooth systems.

6.1. Codimension-one bifurcations of equilibria

6.1.1. B1: The boundary equilibrium bifurcation

From (16) we have two possible scenarios:

- *Persistence*, where an equilibrium turns into a pseudoequilibrium.
- *Nonsmooth Fold*, where an equilibrium and a pseudoequilibrium meet at the boundary and annihilate.

These are illustrated in Fig. 12, and analytic conditions that discriminate between the two scenarios can be derived from the local geometry as follows. Assume that the vector fields and the switching boundary depend on a real parameter γ, so $f_1 = f_1(\mathbf{x};\gamma)$, $f_2 = f_2(\mathbf{x};\gamma)$, $h = h(\mathbf{x};\gamma)$. Then assume, without loss of generality, that f_1 has an equilibrium at $\mathbf{x} = 0$, undergoing a boundary equilibrium bifurcation when $\gamma = 0$, and that $h > 0$ in R_1 and $h < 0$ in R_2. Writing the Jacobian matrix of f_i as $f_{i,\mathbf{x}}$, we have that:

- if $h_{,\mathbf{x}}(0;0)[f_{1,\mathbf{x}}(0;0)]^{-1}f_2(0;0) > 0$, there is persistence at $\gamma = 0$,
- if $h_{,\mathbf{x}}(0;0)[f_{1,\mathbf{x}}(0;0)]^{-1}f_2(0;0) < 0$, there is a nonsmooth fold at $\gamma = 0$.

These conditions are derived as follows. For nonzero γ, a point $\bar{\mathbf{x}} \in \Sigma$ is a pseudoequilibrium if it satisfies (16) with $\mu < 0$ and it lies on Σ, hence it satisfies:

$$\begin{aligned} &f_1(\bar{\mathbf{x}},\gamma) = \mu f_2(\bar{\mathbf{x}},\gamma), \quad \mu < 0,\\ &h(\bar{\mathbf{x}},\gamma) = 0. \end{aligned} \tag{17}$$

Near the origin and for small γ, we can approximate f_i and h by $f_1 \approx f_{1,\mathbf{x}}(0;0)\mathbf{x} + f_{1,\gamma}(0;0)\gamma$, $f_2 \approx f_2(0;0)$, and $h \approx h_{,\mathbf{x}}(0;0)\mathbf{x} + h_{,\gamma}(0;0)\gamma$. The quantity $f_{1,\gamma}(0;0)\gamma$ is identically zero since $f_1(0)$ is constant. Then, (17) can be simplified to

$$\begin{aligned} &f_{1,\mathbf{x}}(0;0)\mathbf{x} = \mu f_2(0;0), \quad \mu < 0,\\ &h_{,\mathbf{x}}(0;0)\mathbf{x} = -h_{,\gamma}(0;0)\gamma, \end{aligned} \tag{18}$$

whose solution is

$$\begin{aligned} &\mathbf{x} = \mu[f_{1,\mathbf{x}}(0;0)]^{-1}f_2(0;0),\\ &\mu = \frac{-h_{,\gamma}(0;0)\gamma}{h_{,\mathbf{x}}(0;0)[f_{1,\mathbf{x}}(0;0)]^{-1}f_2(0;0)}. \end{aligned} \tag{19}$$

The genericity conditions B1 in Section 5.1.1 ensure that these expressions are well defined. The pseudoequilibrium exists only when μ in (19) is negative, while for our choice of h the equilibrium exists (belongs to R_1) only when $h(0,\gamma) \simeq h_{,\gamma}(0;0)\gamma > 0$, giving the conditions as above.

Example. Let

$$\begin{aligned} &f_1 = (x_1 + kx_2, x_1),\\ &f_2 = (0, 1), \end{aligned} \tag{20}$$

and $h = x_2 + \gamma$, then $k > 0$ gives persistence, while $k < 0$ gives a nonsmooth fold, as γ passes through zero. Applying (9) gives $f_s = (x_1 - k\gamma, 0)/(1 - x_1)$ (see Fig. 13).

The classification into persistence and nonsmooth fold cases can alternatively be obtained algebraically, by linearizing the vector fields f_1 and f_s about the boundary equilibrium point, and considering the characteristic polynomials of their respective Jacobians. Such analysis can be found in [71,72], where Feigin's classification [73] for fixed points of piecewise smooth maps was extended to flows.

The distinction into persistence/nonsmooth fold cases above does not give a full account of the nearby dynamics, and indeed none is known. For planar systems, it is known that branches of limit cycles (and even chaotic attractors) can emerge from boundary equilibrium bifurcation points, see Section 7.2, though at present there are no tools known for generalizing these results to n dimensions.

6.1.2. B2: a pseudoequilibrium traversing a two-fold

No study of bifurcations involving pseudoequilibria passing between sliding and escaping regions at two-folds have been made, to our knowledge, in $n > 3$ dimensions (reference is made in [18] to a paper [74], however, that applies point mapping techniques to study two-folds in higher dimensions). The leading order dynamics near a two-fold in three dimensions was studied in [18,23,65], where the codimension-one bifurcation of a pseudoequilibrium, passing between sliding and escaping regions via a two-fold, is a consequence of higher order analysis briefly introduced in [65], with a deeper analysis forming the subject of [63]. An illustrative example taken from that paper is shown in Fig. 14, where a branch of non-sliding limit cycles appears as a pseudoequilibrium traverses the two-fold. For n dimensional systems, the bifurcation B2 takes place when f_1 and f_2 are antiparallel at some point on the two-fold, but little else is known.

6.2. Codimension-one bifurcations of limit cycles

In a smooth vector field, the codimension-one bifurcations that can affect limit cycles have been shown to be few in number and are well understood (see e.g. [1]). This success appears unlikely to

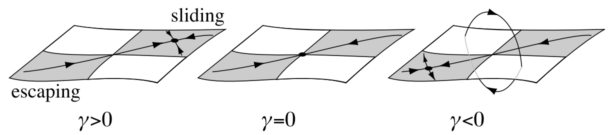

Fig. 14. The passage of a pseudoequilibrium through a two-fold in three dimensions. A pseudonode in the sliding region becomes a pseudosaddle in the escaping region. A non-sliding limit cycle (existing either for $\gamma > 0$ or $\gamma < 0$) is also created in the bifurcation. The bifurcation parameter γ is the quantity $1-(\mathcal{L}_{f_1}\mathcal{L}_{f_2}h)(\mathcal{L}_{f_2}\mathcal{L}_{f_1}h)(\mathcal{L}_{f_1}\mathcal{L}_{f_1}h)^{-1}(\mathcal{L}_{f_2}\mathcal{L}_{f_2}h)^{-1}$ evaluated at the two-fold, and vanishes when f_1 and f_2 are antiparallel there. Such a cycle is shown for $\gamma < 0$. A second scenario is obtained by reversing the direction of time.

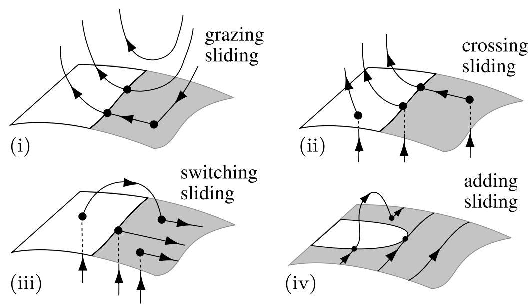

Fig. 15. The four sliding bifurcations. Three of these take place when a non-sliding segment hits a fold, and are called: grazing, crossing, and switching–sliding bifurcations. The fourth takes place when a sliding segment hits a cusp, and is called an adding–sliding bifurcation. A continuous change of initial condition gives a continuous change in the orbit topology, and in the orbit's interaction with the switching boundary.

be replicated in piecewise smooth systems, as is already clear from the many codimension-one bifurcations known in two-dimensions (see Section 7.1). To make headway into the general study of global discontinuity-induced bifurcations, new tools are likely to be needed. At the moment, the most promising approaches are based on a local analysis at the switching boundary, and exploit the fact that discontinuity-induced bifurcations correspond to a non-generic configuration of segments, specifically, in the case of a limit cycle, at points where the flow is tangent to the switching boundary.

As we saw in Section 5.1.2, a cycle undergoes a sliding bifurcation when one of its segments reaches a fold (B3), a cusp (B4), or a two-fold (B5). A classification of the possible bifurcation scenarios is made possible by analysing the geometry of the flow near these points. In [59], all structurally stable configurations of orbits originating from a continuous one-parameter set of initial conditions near a fold, cusp, or two-fold, are identified. These classify the local flow into a small set of topological classes based on the type of tangency. Figs. 15 and 16 portray these configurations. Eight dual scenarios, describing families of orbits terminating at a continuous one-parameter set of final conditions, are obtained by reversing arrows in the figures. The configuration in Figs. 15(i)–(iii) and 16(i) take place at a fold, Fig. 15(iv) at a cusp, and Fig. 16(ii)–(iv) at a two-fold.

To link these portraits with the orbit geometry of bifurcating cycles, it suffices to decompose the Poincaré map for the bifurcating cycle into local and a global parts as follows. Take a small neighbourhood B of the tangency point (see Fig. 17), and a Poincaré section on the boundary of B, then the Poincaré map can be written as the composition of two maps, $\beta(\mathbf{x},\gamma)$ describing the flow in B, and $\phi(\mathbf{x},\gamma)$, describing the flow outside B. Both maps depend on the bifurcation parameter γ. Consider a cycle satisfying one of the bifurcation conditions in (B3), (B4) or (B5), for $\gamma=\bar{\gamma}$, and call $\bar{\mathbf{x}}$ an intersection of the cycle with the boundary of B. Existence of the cycle ensures that the image of the couple $(\bar{\mathbf{x}},\bar{\gamma})$

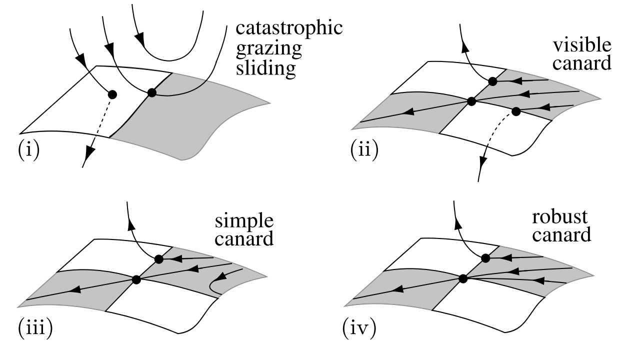

Fig. 16. The four catastrophic sliding bifurcations. These include one case that occurs when a non-sliding segment hits a fold, called a catastrophic grazing–sliding bifurcation. The others take place when a sliding segment hits a two-fold, and are called visible, simple, and robust canards.

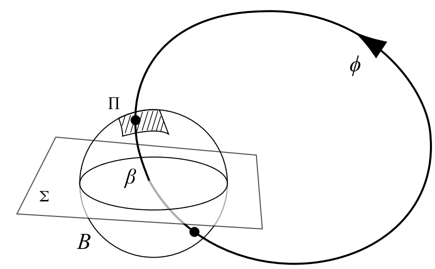

Fig. 17. Local analysis of a limit cycle at a discontinuity. Take a neighbourhood B of the cycle's intersection with the switching boundary Σ, and a Poincaré map on the section Π, then decompose the map into a local part β inside B, and a global part ϕ outside.

under one of four, possibly set-valued, functions, given by

$$\phi(\beta(\mathbf{x},\gamma),\gamma)-\mathbf{x}, \tag{21}$$

$$\phi^{-1}(\beta^{-1}(\mathbf{x},\gamma),\gamma)-\mathbf{x}, \tag{22}$$

$$\phi(\mathbf{x},\gamma)-\beta^{-1}(\mathbf{x},\gamma), \tag{23}$$

$$\phi^{-1}(\mathbf{x},\gamma)-\beta(\mathbf{x},\gamma), \tag{24}$$

always contains zero. Then the existence of a family of cycles can be discussed with the help of the Implicit Function Theorem whenever one of the functions (21)–(24) is smooth and single-valued. The details of this reasoning are summarized in Section 6.2.2, here we summarize the basic results as follows:

- If (21) or (22) is single-valued at $(\mathbf{x},\gamma)=(\bar{\mathbf{x}},\bar{\gamma})$, then the bifurcating cycle belongs to a family parametrized by γ, with geometry near the tangency as in Fig. 15. Notice that the two branches of the family originating at $\gamma=\bar{\gamma}$ may exist on either side of $\bar{\gamma}$, or on the same side. This produces persistence and nonsmooth fold scenarios, as explained in Section 6.2.1.
- If (21) and (22) are set-valued at $(\mathbf{x},\gamma)=(\bar{\mathbf{x}},\bar{\gamma})$, but (23) or (24) is single-valued, then either a one-parameter family of cycles coexists for $\gamma=\bar{\gamma}$, with geometry as in Fig. 15, or the bifurcating cycle disappears as γ is changed, following the scenario in Fig. 16(i). When these conditions hold, orbits near the bifurcating cycle contain both sliding and escaping segments.
- If (21)–(24) are all set-valued at $(\mathbf{x},\gamma)=(\bar{\mathbf{x}},\bar{\gamma})$, then either ϕ and ϕ^{-1} are both set-valued, and no local analysis can be done, or β and β^{-1} are both set-valued, which means that the cycle is touching a two-fold, following one of the scenarios in Fig. 16(ii)–(iv).

One should note that the portraits in Figs. 15 and 16 can be considered as showing the local geometry of sets other than limit cycles, such as one-dimensional stable manifolds [75], one

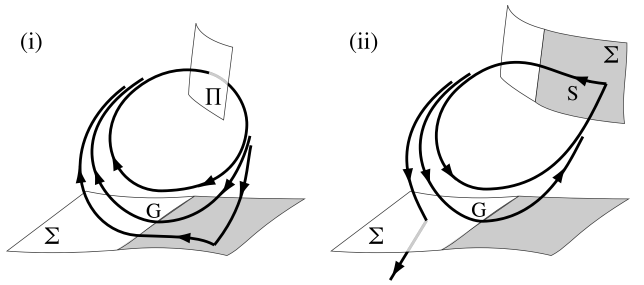

Fig. 18. Examples of: (i) a persistent grazing–sliding bifurcation, and (ii) a catastrophic grazing–sliding bifurcation. In (i), a cycle gains a sliding segment, and the cycle is smooth away from Σ, so it can be described by an invertible return map to the Poincaré section Π. In (ii), a cycle is abruptly destroyed, and the cycle has a sliding segment S, which lies away from the grazing point G where the bifurcation occurs, therefore it has no invertible return map.

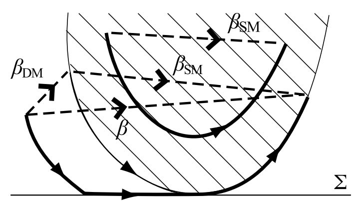

Fig. 19. Near a tangency, the state space is divided into two regions with different orbit topologies, depending on their intersection with Σ. In the hatched region the flow induces a map $\beta = \beta_{SM}$, while elsewhere the flow induces a map $\beta = \beta_{SM} \circ \beta_{DM}$.

dimensional separatrices, etc., though such extensions are beyond the scope of this section.

The key observation here, that study of discontinuity-induced bifurcations of a limit cycle can be reduced to local geometry, is attributable to Nordmark's pioneering work on grazing bifurcations in impact and friction oscillators (see [76,28]). An extension to sliding bifurcations was presented in [66], where maps are derived that correct for the presence of the discontinuity when applied to a smooth flow continued from one side of the switching boundary. These *discontinuity maps* are, however, limited to cases where the cycle encounters no more than one sliding or escaping region, as in Fig. 18(i). In such cases they allow us to determine the differentiability of the Poincaré map, and hence determine what kinds of global bifurcations the cases in Fig. 15 will give rise to. The effect of encountering a second sliding or escaping region opens up the possibility of the cases in Fig. 16. For example, Fig. 18(ii) shows an instance of the catastrophic grazing–sliding bifurcation from Fig. 16(i) when a cycle encounters both sliding and escaping regions.

6.2.1. B3–B4: discontinuity maps at a fold or cusp

Consider the configurations in Fig. 15. In all scenarios, as noted above, the neighbourhood B can be divided into two parts where the sequence of segments composing an orbit takes a qualitatively different form, such as those hatched and unhatched in Fig. 19. Consequently, the map β takes two different functional forms in the two regions. In the literature, and as illustrated in Fig. 19, it is common therefore to decompose β into a smooth part, β_{SM}, which is the same in both regions, and a piecewise-smooth part, β_{DM}, which is the identity on one region (shaded in Fig. 19). Then $\beta = \beta_{SM} \circ \beta_{DM}$, and β_{DM} is the discontinuity map.

If ϕ and β or ϕ^{-1} and β^{-1} are single-valued, the analysis of the dynamics near the bifurcating cycle can be reduced to the analysis of a Poincaré map $\phi \circ \beta$ or $\beta^{-1} \circ \phi^{-1}$. These are the cases associated with the scenarios in Fig. 15. If ϕ is also invertible, the differentiability of the Poincaré map depends only on the properties of β, and thus of β_{DM}. This may not be the case if ϕ is not invertible, as the following example shows.

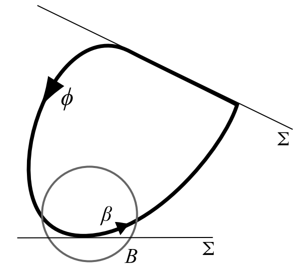

Fig. 20. The Poincaré map of a cycle undergoing a sliding bifurcation is constant if the cycle has a sliding segment, regardless of the discontinuity in the local map β.

Example. Consider a system with the state portrait depicted in Fig. 20. In this case the map $\phi(\mathbf{x})$, describing the dynamics outside B, has constant value for all $\mathbf{x}$ due to the sliding segment. Hence the one-dimensional Poincaré map of the cycle is constant, regardless of the form of map β.

Normal forms for the discontinuity maps of the scenarios in Fig. 15 are found in the literature. A full review is found in [17], and here we review only their analytic form. In the crossing–sliding and switching–sliding bifurcations, we call $\mathbf{x}^*$ the point where the non-sliding segment of a periodic orbit hits the switching manifold. In the grazing–sliding bifurcation, we call $\mathbf{x}_{\min}$ the point along a non-sliding segment of a periodic orbit (eventually continued beyond the switching manifold) that minimizes the function h. In the adding–sliding bifurcation, we call $\mathbf{x}_{\min}$ the point along a non-sliding segment of a periodic orbit (eventually continued beyond the fold) that minimizes the function $\mathcal{L}_{f_1}h$. In all scenarios we assume that f_1 can be continued to $h(\mathbf{x}) \leq 0$. Then, the Taylor expansions of the (zero-time) discontinuity maps β_{DM} for the four generic sliding bifurcations (in Fig. 15) can be written, to leading order (omitting arguments), in the form:

- grazing–sliding,

$$\beta_{DM} : \mathbf{x} \mapsto \begin{cases} \mathbf{x} & \text{if } \sigma \geq 0, \\ \mathbf{x} + \dfrac{(f_2 - f_1)\,h}{\mathcal{L}_{f_2}h} & \text{if } \sigma < 0, \end{cases} \tag{25}$$

where $\sigma(\mathbf{x}) = h(\mathbf{x}_{\min})$;
- crossing–sliding,

$$\beta_{DM} : \mathbf{x} \mapsto \begin{cases} \mathbf{x} & \text{if } \sigma \geq 0, \\ \mathbf{x} + \left(\mathcal{L}_{f_1}h\right)^2 \dfrac{(f_2 - f_1)}{2\mathcal{L}_{f_2}h\mathcal{L}_{f_1}^2 h} & \text{if } \sigma < 0, \end{cases} \tag{26}$$

where $\sigma(\mathbf{x}) = \mathcal{L}_{f_1}h(\mathbf{x}^*)$;
- switching–sliding,

$$\beta_{DM} : \mathbf{x} \mapsto \begin{cases} \mathbf{x} & \text{if } \sigma \geq 0, \\ \mathbf{x} + \dfrac{2(\mathcal{L}_{f_1}h)^3}{3(\mathcal{L}_{f_2}h)^2(\mathcal{L}_{f_1}^2 h)^2}Q & \text{if } \sigma < 0, \end{cases} \tag{27}$$

where $\sigma(\mathbf{x}) = -\mathcal{L}_{f_1}h(\mathbf{x}^*)$;
- adding–sliding,

$$\beta_{DM} : \mathbf{x} \mapsto \begin{cases} \mathbf{x} & \text{if } \sigma \geq 0, \\ \mathbf{x} - \dfrac{9(\mathcal{L}_{f_1}h)^2}{2(\mathcal{L}_{f_2}h)^2\mathcal{L}_{f_1}^3 h}Q & \text{if } \sigma < 0, \end{cases} \tag{28}$$

where $\sigma(\mathbf{x}) = \mathcal{L}_{f_1}h(\mathbf{x}_{\min})$;

and where

$$Q = q\mathcal{L}_{f_2}h + (f_1 - f_2)\mathcal{L}_q h, \qquad q = f_{1,\mathbf{x}}f_2 - f_{2,\mathbf{x}}f_1.$$

These discontinuity maps were derived for the first time in [66]. For cases obtained by reversing time in Fig. 15, where a cycle has a segment in the escaping region, the (set-valued) discontinuity map can be easily deduced. These discontinuity maps have been shown to be differentiable in all except the grazing–sliding case, where the map is piecewise smooth but continuous at the switching boundary. As a consequence, if a Poincaré map is well defined, it is smooth in all scenarios in Fig. 15 except grazing–sliding. When the Poincaré map is smooth, the only expected topological effect of the bifurcation is to change the number and type of segments constituting the cycle. When the map is piecewise smooth and continuous, then persistence, nonsmooth fold, and nonsmooth period doubling scenarios are possible, and other invariant sets can be involved in the bifurcation. The theory of bifurcations of piecewise smooth continuous maps is as young and incomplete as that of piecewise smooth flows, but some results are to be found in the literature, for example in [77].

6.2.2. Existence of cycles

Here we explain in detail how the functions (21)–(24) can be used to obtain the results of the previous section. Consider a cycle O that, for a particular value $\bar{\gamma}$ of the bifurcation parameter γ, undergoes a bifurcation as in scenarios B3, B4 or B5 (that is, it touches a fold, a cusp, or a two-fold). The Poincaré map of the cycle can be written as a composition of a local part $\beta(\mathbf{x}, \gamma)$ and a global part $\phi(\mathbf{x}, \gamma)$, as in Fig. 17. Both maps depend on the bifurcation parameter γ and, if either ϕ or β is set-valued, then so is the Poincaré map. Genericity conditions for B3, B4, and B5 require that all intersections away from the neighbourhood B be transversal, and for B3 and B4, they require also that the cycle be hyperbolic (this is not necessary in the case of a two-fold, where the Poincaré map and its inverse are set-valued and hyperbolicity has no meaning). These conditions ensure that, if ϕ or ϕ^{-1} is single-valued, then it is smooth, since all intersections are transversal. Similarly, if β or β^{-1} is single-valued, then it can be expressed as the composition of one or two smooth functions, expressing the flow near a tangency. Hence a composition of ϕ, β and their inverses gives a function that is piecewise smooth, composed of two smooth parts continuously adjoint.

Now consider the four functions (21)–(24). We have either that one of these functions is single-valued near $(\bar{\mathbf{x}}, \bar{\gamma})$, or both β and β^{-1} are set-valued. Moreover, by the considerations above, if any one of (21)–(24) is single-valued, then $(\bar{\mathbf{x}}, \bar{\gamma})$ is in the zero set of both its smooth parts. The Implicit Function Theorem can be applied to each of the two smooth parts, smoothly extended in a neighbourhood of $(\bar{\mathbf{x}}, \bar{\gamma})$.

If either (21) or (22) is single-valued, then the hyperbolicity of the cycle O at $\bar{\boldsymbol{\gamma}}$ implies that the Jacobian of (21) or (22) in $\mathbf{x}$ is nonsingular. By the Implicit Function Theorem, O sits at the intersection of two families of solutions, one for each smooth part of β. The two families adjoin continuously, but can be defined for values of γ on the same side of $\bar{\gamma}$, or on opposite sides. This gives nonsmooth fold and persistence scenarios, as we see in the next section. Since the path of $\mathbf{x}(\gamma)$ is continuous, the set of orbits beginning or terminating at $\mathbf{x}(\gamma)$ has geometry near the tangency as in Fig. 15. The direction of time is as in the figure if (21) is single-valued, while time is reversed if (22) is single-valued.

If both (21) and (22) are set-valued, but (23) or (24) are single-valued, then the cycle O has at least an escaping segment and is touching the border of a sliding region in B, or it has a sliding segment and touches a border of an escaping region in B. The Jacobians in $\mathbf{x}$ of both ϕ or ϕ^{-1} and β^{-1} or β have a null space that is, typically, one dimensional in the presence of sliding and escaping segments. Unless these null spaces are orthogonal, the Jacobian of (23) or (24) has a one-dimensional null space, which means that either no solution exists for γ near $\bar{\gamma}$, or a one-parameter family of solutions coexists at $\gamma = \bar{\gamma}$. The first case corresponds to the scenario in Fig. 16(i). In the second case, a one-parameter family of cycles has local geometry as in Fig. 15.

Finally, if none of (21)–(24) is single-valued, the Implicit Function Theorem cannot be used.

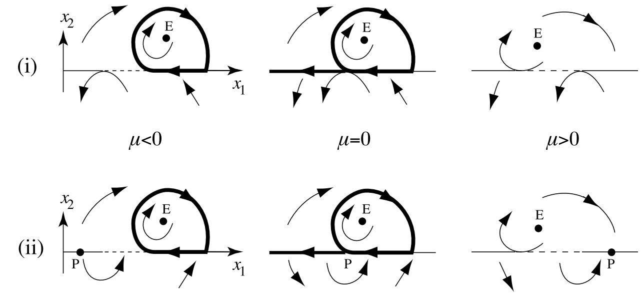

Fig. 21. Example of a limit cycle destroyed in (i) a visible canard case with the '+' sign from (29), and (ii) a simple canard case with the '−' sign from (29), of catastrophic sliding bifurcation in two dimensions. Two folds swap position as μ changes sign. When $\mu = 0$, the sliding and escaping regions are connected via an orbit which passes through a degenerate-fold, and is called the 'canard' orbit. Both of these have recently been found to play important roles during the "canard explosion" in a singularly perturbed van der Pol oscillator, see [78], and their deeper connection to singular perturbation phenomena is the subject of ongoing study. Equilibria and pseudoequilibria are marked E and P.

6.2.3. B5: canards at a two-fold

Regarding the cases of catastrophic sliding bifurcations referred to as *canards* in Fig. 16(ii)–(iv), nothing is known other than the local geometry required for them to occur. No tools yet exist to analyse them globally, but they have been shown in [78] to be related to canards in singularly perturbed systems. This connection is briefly discussed in Section 8.3, and motivates the terminology applied to them in Fig. 16(ii)–(iv). These 'nonsmooth canards' effectively provide a classification of the different local forms that 'canard explosions' (to use the terminology from singular perturbation theory) can take.

Example. Consider the system in Fig. 21, with a switching boundary $h = x_2$, and where

$$\begin{aligned} f_1 &= (x_1 + \omega x_2 - \omega - \omega^{-1}, x_2 - \omega x_1) \\ f_2 &= (\pm\omega^{-1}, x_1 - \mu), \end{aligned} \tag{29}$$

where $\omega = 10$. For $\mu > 0$, a limit cycle with a sliding segment exists (see Fig. 21). There are two folds, at $x = (0, 0)$ and $x = (\mu, 0)$, which coincide when $\mu = 0$ so that an orbit passes from sliding to escaping; we call this orbit the *canard*. The tangency in f_1 is visible, while the tangency in f_2 is invisible or visible taking respectively the $+$ or $-$ signs in (29). These are depicted in Fig. 21. For $\mu < 0$, in (i) all orbits are attracted to a pseudoequilibrium (P) in the sliding region, and in (ii) all orbits end up in R_2; in each case the limit cycle has vanished via a simple (i) or visible (ii) canard case of catastrophic sliding bifurcation. The fold that exists at $\mu = 0$ is non-generic, and were we to add a third dimension, it would generically unfold to form a two-fold. From this association we see that this is just an example of the two-dimensional analogue of the visible canard classified in Fig. 16.

6.3. Codimension-two bifurcations of equilibria

We are not aware of any analysis of bifurcations of type B1.2–B1.4 from Section 5.2.1 that apply in a general number of dimensions (results limited to two dimensions are discussed in Section 7.1), but some results exist regarding type B1.1. Nonhyperbolic equilibria generically undergo a Hopf or saddle–node bifurcation under one parameter variation. The case of a boundary equilibrium bifurcation simultaneous to a saddle–node of the pseudoequilibrium has been recently studied in [64]. The local unfolding exhibits a boundary-equilibrium bifurcation changing from persistence to nonsmooth fold across the codimension two point, and a saddle–node of pseudoequilibria adjoining the point tangentially to the boundary equilibrium. The dual case, of a boundary equilibrium bifurcation simultaneous to a saddle–node bifurcation of the

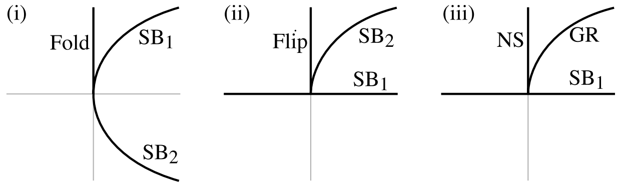

Fig. 22. Partial unfolding of the codimension-two sliding bifurcations of nonhyperbolic limit cycles. (i): sliding bifurcation + fold; SB_1 and SB_2 are sliding bifurcations the two cycles involved in the fold bifurcation. (ii): sliding bifurcation + flip; SB_1 and SB_2 are the sliding bifurcations of the period-one and period-two cycle respectively. (iii) sliding bifurcation + Neimark–Sacker; SB_1 is the sliding bifurcation of the cycle, NS is the Neimark–Sacker, and GR is a grazing bifurcation of the torus. Bifurcation curves near any of these codimension-two points must be arranged as in one of the three portraits. However, the presence of other bifurcation curves, and the dynamics across the curves, depend on the type of sliding bifurcation, and the complete unfoldings are still unknown.

equilibrium, has not been studied, but the analysis of the same bifurcation in the discrete-time case conducted in [79] suggests that a similar unfolding should be expected. All other combinations are still waiting to be explored.

6.4. Codimension-two bifurcations of limit cycles

Sliding bifurcations of Type I in Section 5.2.2 occur when the tangency between the vector fields and the boundary is degenerate. Referring to the genericity conditions in Section 5.1.2, this can be a cusp point or a two-fold in case B3, and a fourth-order tangency ($\mathcal{L}^3_{f_1}h = 0$) or a two-fold in case B4. The case of a grazing with a cusp point or a fourth-order tangency is discussed in [80], where the discontinuity-induced bifurcations that adjoin these codimension-two points are listed, but only the unfolding of the degenerate crossing–sliding is reported, while the unfolding of all other cases is left undone. The case of other possible degeneracies, which involve grazing at two-fold points, have not been analysed to date.

A partial unfolding of all type II bifurcations is provided in [79], for nonhyperbolic cycles undergoing a grazing with a boundary, without further assumptions on the dynamics on the other side of the boundary or on the geometry of the boundary. Intuitively, these codimension-two points are the intersection of (at least) a sliding bifurcation curve and a flip, fold, or Neimark–Sacker bifurcation curve. The analysis in [79] retrieves the local geometry of these curves near the codimension-two point (reported in Fig. 22) for any of the sliding bifurcations admitted by the configurations in Figs. 15 and 16, but it does not allow deduction of the complete set of bifurcation curves, nor of the type of dynamics that surround these curves. The complete unfolding of a single case, the fold case of grazing–sliding, can however be deduced from the results in [81] on the fold border-collision in a piecewise continuous map, since the Poincaré map of a grazing–sliding cycle is indeed piecewise smooth continuous. Interestingly, the onset of chaos after the grazing is predicted.

A general analysis of Type III bifurcations has never been carried out.

Finally, some codimension-two bifurcations of n-dimensional piecewise continuous maps have been studied in [77,81,82], and the results should apply to codimension-two grazing–sliding bifurcations of cycles which, as seen in Section 5.1.2, induce a piecewise smooth continuous Poincaré map.

6.5. Boundary-intersection crossing bifurcations

As we discussed in Section 5.3, the most well studied boundary-intersection bifurcations occur when a non-sliding segment of a limit cycle reaches a boundary intersection. Local discontinuity maps provide useful insights, just as they did in the absence of boundary intersections. These maps have been obtained in [83,17] in the absence of any sliding/escaping region near the boundary intersection, and have been extended in the case of a single sliding region, and provided no sliding occurs along the intersection [84]. In all cases, the discontinuity map has been found to be piecewise smooth and continuous. This implies that, when a Poincaré map can be defined, it is piecewise smooth continuous. The theory of piecewise smooth continuous maps, discussed in [85,86,20,73,87,88], predicts that the cycle can undergo nonsmooth fold, persistence, and nonsmooth period doubling scenarios analogous to those at a grazing–sliding bifurcation.

7. Specific results in $\mathbb{R}^n$ for $n \leq 3$

In this section, we consider a few examples of phenomena that are unique to piecewise smooth systems, but which have to date eluded generalization to n-dimensions. All of these results are obvious starting points for future work.

7.1. Planar Filippov systems

An extensive study of one parameter bifurcations in planar Filippov systems is made by Kuznetsov et al. in [60]. Here the authors consider a Filippov system where the switching boundary Σ is simply a smooth curve. Depending on certain conditions satisfied by the vector fields, there are four types of special sliding points: singular sliding points, pseudoequilibria, boundary equilibria and tangent points, which we will denote by T.

The approach taken in [60] is to propose different bifurcation scenarios, and examine their topological detail. The scenarios considered are stated to be generic, and assigned normal forms, and a forthcoming paper [89] is intended to prove their genericity. The different scenarios considered are:

1. codimension 1 local bifurcations
   - collisions of a focus with Σ (*boundary focus*)
   - collisions of a node with Σ (*boundary node*)
   - collisions of a saddle with Σ (*boundary saddle*)
   - collisions of tangent points
   - collisions of pseudoequilibria
2. codimension 1 global bifurcations
   - bifurcations of sliding cycles
   - pseudo-homoclinic bifurcations
   - pseudo-heteroclinic bifurcations
3. codimension 2 local bifurcations
   - degenerate boundary focus
   - boundary Hopf
4. codimension 2 global bifurcations
   - grazing–sliding of a nonhyperbolic cycle.

The constructive approach taken to derive these has its limitations, even in two dimensions. In the case of codimension 1 global bifurcations, the authors in [60] considered bifurcations of cycles which collide with Σ. They uncover the four familiar sliding bifurcations of grazing–sliding, adding–sliding, switching–sliding and crossing–sliding (Fig. 15). But they do not find the catastrophic sliding bifurcations (Fig. 16) described in [59], explicit planar examples of which are shown here in Figs. 21 and 25; these also are not discussed generally in a forthcoming paper [89], but a case similar to the ‘simple canard’ in Fig. 21 appears in [89] as part of the codimension two unfolding of a fold-cusp singularity. A direct inspection of orbit configurations on either side of the switching boundary in Fig. 21 reveals that these bifurcations must be present in the problem in general, hence it is legitimate to ask whether, in the other cases considered in [60], the constructive approach may have left other codimension-one bifurcations undiscovered.

At present it is not known to what extent this work can be generalized, either to systems with boundary intersections, or with n-dimensions. An obvious generalization is to a bimodal (two-region) system in n-dimensions. In that case, the switching boundary has $n - 1$ dimensions. Equilibria close to Σ can then take a large number of different forms, depending on the eigenvalues of the piecewise smooth vector field's Jacobian. It is not immediately clear how any of these generalizations should be approached. The classification of sliding bifurcations in Figs. 15–16 applies in n-dimensions (see [59]) and therefore partially addresses the problem, but only away from boundary intersections and equilibria. The incompleteness of the classification of planar codimension-one bifurcations proposed in [60] reveals that piecewise smooth systems have surprises to spare. While completing this remains an open problem even in two dimensions, a more pressing need is to know how current results can be applied in higher dimensions.

7.2. Generalized Hopf bifurcation

For a smooth system, a Hopf bifurcation occurs when a complex conjugate pair of eigenvalues crosses the imaginary axis. This cannot occur at a switching boundary because of the impossibility of linearizing the vector field across the discontinuity at the origin. Strictly speaking, therefore, a Hopf bifurcation in a piecewise smooth system can only occur if contained entirely within one of the open regions R_i.

In this section we consider periodic orbits that bifurcate from stationary solutions of Filippov systems. These have been called *generalized Hopf bifurcations* [90–92] or discontinuity-induced Hopf bifurcations [17,72]. The basic idea here has been to characterize the generalized Hopf bifurcation as being given by the change from stable focus to unstable focus via a centre in a basic piecewise linear system, which takes the place of linearization in the smooth case.

Again little work has been done to generalize these results to arbitrary dimensions. Notable exceptions are restricted to the case where the vector field is continuous across the switching surface. For piecewise smooth continuous systems, [93] extends the idea of an invariant manifold to describe invariant cones for generalized Hopf bifurcations, and Simpson and Meiss [92] prove the following result:

Theorem 7.1 ([92]). *Suppose that the system is continuous and sufficiently differentiable, and that it has an equilibrium that crosses the switching boundary at $x = x^*$ when $\mu = 0$. As $\mu \to 0_+$ the eigenvalues of the equilibrium approach $\lambda_L \pm i\omega_L$, and as $\mu \to 0_-$ they approach $-\lambda_R \pm i\omega_R$ where $\lambda_L, \lambda_R, \omega_L, \omega_R > 0$. Define $\Lambda = \frac{\lambda_L}{\omega_L} - \frac{\lambda_R}{\omega_R}$. Then if $\Lambda < 0$, there exists $\epsilon > 0$ such that for all $\mu \in (0, \epsilon)$ there is an attracting periodic orbit whose radius is $\mathcal{O}(\mu)$ away from x^* and $\forall \mu \in (-\epsilon, 0)$, there are no periodic orbits near x^*. If $\Lambda > 0$, $\exists \epsilon > 0$ such that $\forall \mu \in (-\epsilon, 0)$ there is a repelling periodic orbit whose radius is $\mathcal{O}(\mu)$ away from x^* and $\forall \mu \in (0, \epsilon)$, there are no periodic orbits near x^*.*

The same authors emphasize the difference between this and the smooth Hopf bifurcation. First, the generalized Hopf solution is made up of two spiral segments; it is not elliptical. Second, the growth rate is linear in μ, whereas it is $\mathcal{O}\left(\mu^{1/2}\right)$ in the smooth case. Finally the criticality of the bifurcation is determined by the linear term Λ, rather than by cubic terms.

A paper by Han and Zhang [94] studies the different ways in which planar limit cycles can be produced from the three different possible piecewise smooth foci. They base their results on previous results of [95], but as before, the possibility of generalization to higher dimensions is not obvious.

Simpson and Meiss [92] speculate why it might be difficult to generalize their work to n-dimensions. The main challenge is that it is not clear how to obtain the required centre manifold reduction. They also point out the important fact that, in higher dimensions, an equilibrium on a switching boundary can be unstable even when both Jacobians have all their eigenvalues in the left hand plane, citing an example given in [96].

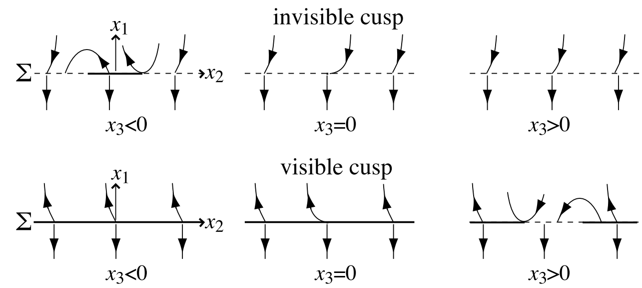

Fig. 23. Double tangencies and sliding regions: the cusp in two dimensions. As two tangencies on one side of Σ collide in a cusp, a region opens up between them. Regions of crossing (dashed line) surround a region of escaping (full lines) in the case of an invisible cusp, and a regions of escaping surround a region of crossing in the case of a visible cusp. Reversing arrows changes escaping to sliding.

7.3. Bifurcations of the sliding boundaries

Section 3 provides a short overview on the geometry of the boundaries of crossing, sliding, and escaping regions Σ. Clearly, a perturbation of a system's equations can in general alter this geometry, therefore it is natural to consider some obvious topological changes that occur at the boundaries: either at self-intersections of the switching boundary Σ, or at the sliding boundaries on Σ. Bifurcations concerning the geometry of boundary intersections are mostly ignored in the literature. Topological changes of sliding boundaries have received more attention, at least in two or three dimensions (see [18,60,23]).

Degenerate tangencies give rise to structurally unstable sliding boundaries, and when perturbed, these produce bifurcations that create regions of sliding, escaping, or crossing. Consider for instance the cusp, which, as noted in Section 3, is topologically stable in three or more dimensions. We can consider (13) in the two dimensional space of (x_1, x_2) and treat x_3 as an unfolding parameter. When $x_3 = 0$, Σ consists of the cusp point surrounded by regions of crossing if the cusp is invisible, and sliding/escaping if the cusp is visible. As x_3 passes through zero, a bifurcation takes place that opens a region between the points $x_2 = \pm\sqrt{x_3}$ on $x_1 = 0$, bounded by folds, one visible and one invisible. The normal form (13) gives sliding if we take the '+' signs and escaping if we take the '−' signs; the latter is illustrated in Fig. 23.

The same reasoning applies in the case of the two-fold. Its two-dimensional counterpart is the "double fold", occurring when the piecewise smooth flow has tangencies above and below Σ, which exchange ordering as a parameter varies, given by a normal form

$$\begin{aligned} f_1 &= \pm(s_1 x_2, 1), \\ f_2 &= \pm(s_2(x_1 + 2x_2 - \mu), 1), \end{aligned} \tag{30}$$

where we can choose the $\pm$ signs on each row independently, where $s_{1,2}$ take values ± 1 and μ is an unfolding parameter. The two folds lie at $x_2 = 0$ and $x_2 = \mu/2$, and their visibility depends on s_1 and s_2 as in (14). As μ changes sign, either a region of crossing closes and re-opens and lies between sliding and escaping regions (shown in Fig. 24 for the case with two visible folds) or a region of sliding closes then a region of escaping opens, lying between crossing regions. In each case a bifurcation has taken place as μ changed sign. (We should remark that in [60] the double fold is given by different normal forms depending on the types of visibility involved. However, (30) provides a single form giving the correct

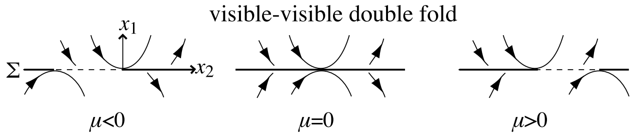

Fig. 24. Double tangencies and sliding regions: two folds in two dimensions. As a pair of tangencies, one each side of Σ, exchange ordering, a crossing region (dashed line) opens up between two visible folds, lying between sliding and escaping regions (full lines). This corresponds to (30) with the '+' signs and with $s_1 = -s_2 = 1$.

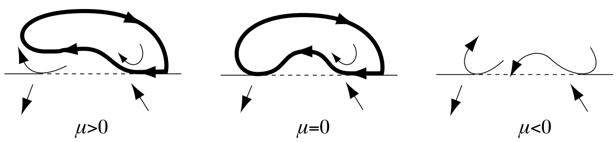

Fig. 25. Example of a periodic orbit destroyed in a catastrophic grazing–sliding bifurcation. As a parameter μ changes we have: (i) a stable periodic orbit with a sliding segment, (ii) the periodic orbit grazes a visible fold at the boundary of an escaping region, (iii) no attractors remain.

topology for all cases, including pseudoequilibria and limit cycles which we do not discuss here.)

Likewise, higher order degeneracies of tangency points will cause bifurcations in two or more dimensions. Teixeira [23] has considered four cases of one-parameter bifurcations of sliding regions in three dimensional systems, called the dovetail (which occurs at a fourth order tangency), the lips and beak-to-beak (which occur at a degenerate form of the cusp), and the degenerate two-fold (where two fold curves intersect non-transversally); each of these will have topologically stable counterparts in four or more dimensions, just as the planar cusp and double fold have stable counterparts in three or more dimensions.

8. Where do we go from here?

In previous sections we have drawn together results that point towards a general theory of discontinuity-induced bifurcations, and efforts to this end are unlikely to have peaked. Pressing concerns involve how to balance generality and rigour in classifying nonsmooth systems, to avoid, for instance, semantic differences obscuring dynamical similarities. When are two nonsmooth systems topologically equivalent, and when do they undergo bifurcations? In spite of the results we have brought together here, these are issues on which a consensus is yet to be reached. In this section we briefly outline some promising directions for future progress.

8.1. Pinching and regularization

One way of dealing with a discontinuity on the righthand side of the equation $\dot{\mathbf{x}} = f$ is to smooth it out. If $f = f^+$ for $h < 0$, and $f = f^-$ for $h > 0$, we can approximate the discontinuity at $h = 0$ by a slope over $|h| < \epsilon$, for some $\epsilon > 0$, by writing

$$\dot{\mathbf{x}} = \frac{1 + \phi(h(\mathbf{x})/\epsilon)}{2} f^+(\mathbf{x}) + \frac{1 - \phi(h(\mathbf{x})/\epsilon)}{2} f^-(\mathbf{x}), \tag{31}$$

where $\phi(y) = -1$ for $y \leq -1$, $\phi(y) = +1$ for $y \geq 1$, and $\phi(y) \in (-1, 1)$ for $y < -1$. Then (31) is called a regularization, and ϕ is a transition function. Further smoothing can be convenient for numerical simulations, giving ϕ a differentiable sigmoidal form instead. Whatever the differentiability of ϕ, it is not known how well regularization approximates the dynamics at a discontinuity.

It has been shown that, given the regularization of a piecewise smooth system, a singularly perturbed system can be found that is topologically equivalent [97], and in particular, that a sliding region is then homoeomorphic to a normally hyperbolic slow manifold. Results so far do not extend to points where a piecewise smooth vector field is tangent to a switching boundary, which are likely to be associated with a loss of hyperbolicity of a slow manifold. Non-hyperbolic points commonly require the introduction of artificial "blow-up" parameters, and their study is ongoing. In [78], however, it was shown by a different method, called pinching, that when slow manifolds are indeed approximated by switching boundaries, their non-hyperbolic points are approximated by two-fold singularities.

Pinching, which can be thought of as a converse to regularization, was introduced in [98] and expanded upon in [78]. Pinching approximates a smooth vector field by a discontinuous one, by collapsing a region of state space to form a switching boundary. Let

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x}), \tag{32}$$

where f and g are smooth functions of the state vector $\mathbf{x} = (x_1, x_2, x_3, \ldots)$. Let ϵ be a positive constant, and let $g(\mathbf{x}) \gg f(\mathbf{x})$ for $|x_1| < \epsilon$, and $f(\mathbf{x}) \gg g(\mathbf{x})$ for $x_1 > \epsilon$. We call $|x_1| < \epsilon$ the pinch zone. We then introduce a new coordinate $y_1 = x_1 - \epsilon \ \mathrm{sign}(x_1)$ over the region $x_1 > \epsilon$, where g is negligible. This defines a new state variable $\mathbf{y} = (y_1, x_2, x_3, \ldots)$, which satisfies

$$\dot{\mathbf{y}} = \begin{cases} f^+(\mathbf{y}), & \text{if } y_1 > 0, \\ f^-(\mathbf{y}), & \text{if } y_1 < 0, \end{cases} \tag{33}$$

where $f^{\pm}(\mathbf{y}) = f(y_1 \pm \epsilon, x_2, \ldots)$, and where $g \ll f$ has been neglected.

The result of pinching is that, at a point $\mathbf{y} = (0, x_2, x_3, \ldots)$ on the switching boundary, we have the differential inclusion

$$\dot{\mathbf{y}} = \{f(\mathbf{z}) + g(\mathbf{z}) : \mathbf{z} = (\xi, x_2, x_3, \ldots),\ \xi \in (-\epsilon, \epsilon)\}. \tag{34}$$

We can approximate the set-valued righthand side by an interpolation between the values of $f + g$ at $\xi = \pm\epsilon$, where g is negligible, resulting in

$$\dot{\mathbf{y}} \approx \{f^+(\mathbf{y}) + (1 - \lambda) f^-(\mathbf{y}) : \lambda \in (0, 1)\} \quad \text{on } y_1 = 0. \tag{35}$$

Then the system (33) with (35) is a Filippov system. Note that we have derived this as an approximation to the smooth system (32), replacing the dynamics in the region $|x_0| < \epsilon$ where g dominates, with dynamics at a switching boundary given by (35).

In [78] it was shown that pinching can be used to study bifurcations in singularly perturbed systems, and is related to the nonstandard analysis [99] approach to studying the highly nonlinear phenomenon of canards. It is hoped that the concepts of pinching and regularization will continue to give insight into the correspondence of singularities and bifurcations between smooth and piecewise smooth systems.

8.2. The notion of a sliding bifurcation

Among the most powerful concepts in bifurcation theory is that of centre manifold reduction, whereby a bifurcation in n-dimensions is reduced to a lower dimensional problem. Very little has been achieved concerning centre manifolds in piecewise smooth systems. As a result, many bifurcations have been described in planar systems, without much discussion of when a given discontinuity-induced bifurcation in n-dimensions can be reduced to a planar problem.

An exception is given by the discontinuity mappings in Section 6.2.1, which classify one-parameter sliding bifurcations using local geometry in lower dimensions, in the neighbourhood of a switching boundary. The implication of Section 6.2 is that many different discontinuity-induced bifurcations can be classified as sliding bifurcations, provided they occur where orbits graze the switching boundary, regardless of the object (such as a periodic orbit or invariant manifold) undergoing the bifurcation. This idea of topological reduction is very different to, though in the spirit of, centre manifold or normal form reduction.

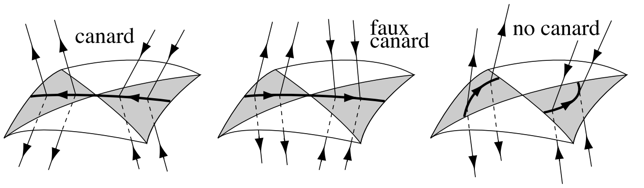

Fig. 26. Canards and two-folds: (i) a canard passes through a two-fold, from sliding to escaping regions; (ii) a faux canard passes from escaping to sliding regions; and (iii) a two-fold without canards. Sliding/escaping regions are shaded, crossing regions are unshaded, and the boundaries between them are folds.

The utility of the topological classification of sliding bifurcations (Figs. 15–16) can be seen by their ability to predict previously unknown global bifurcations. As an example, Fig. 25 (and our earlier Fig. 21) illustrate codimension-one bifurcations of stable limit cycles that have escaped classification until now, but which are easily deduced from the geometry of catastrophic sliding bifurcations. Fig. 25 consists simply of an unstable focus in the upper vector field, with sliding and escaping on the rightmost and leftmost parts of the switching boundary respectively. For $\mu > 0$ a stable limit cycle with a sliding segment encircles an unstable focus. For $\mu = 0$ the cycle grazes the boundary of an escaping region, and a catastrophic grazing–sliding bifurcation takes place. Then, for $\mu < 0$, inspection of the state portrait reveals that no limit cycles can exist, and all orbits eventually end up below the switching boundary.

These bifurcations can equally well occur in higher dimensional systems, or those with more complex switching boundary topologies.

8.3. A pivotal point: the two-fold singularity

Two-fold singularities were proposed in [18,22] to be of fundamental importance if piecewise smooth dynamical systems theory was to venture beyond the plane. Their importance lies in allowing orbits to pass from attracting to repelling regions of state space, that is, from sliding to escaping regions (noting that the attraction/repulsion is strong in the sense that it is not asymptotic, but takes place in finite time). Such orbits are called canards, see Fig. 26.

In [78] it was shown that canards at two-folds (as in Fig. 16) are not only consistent with the canards familiar in singularly perturbed systems [99,100], indeed they can be derived as approximations to them, and furthermore reveal that canard explosions are examples of catastrophic sliding bifurcations.

Canards can have local consequences too, as are revealed at the invisible two-fold, which was dubbed the "Teixeira singularity" in [65]. Since the far reaching work of Filippov [18] and Teixeira [22], the Teixeira singularity has continued to reveal novel local dynamics [35,63,65], and remains a subject of ongoing interest. When the Teixeira singularity exhibits a faux canard, orbits locally wind around the singularity only a finite number of times before entering the sliding region. In [65], a "nonsmooth diabolo" bifurcation was derived, whereby an invariant double cone self-annihilates, and turns faux canards into canards, see Fig. 27. Analysis of higher order terms near the bifurcation, in [63], has shown that, for certain parameters, orbits locally begin and end at the singularity via escaping and sliding segments that lie along a canard. Locally, solutions in the flow therefore visit the singularity recurrently, but become non-unique each time they traverse the two-fold. This means that the trajectory of an orbit leaving the singularity is not determined by how it entered, and the resulting behaviour exhibits a non-deterministic form of chaos. In particular, the study of canards from the piecewise smooth perspective [78] is quite new, and in spite of a proof that Teixeira singularities can

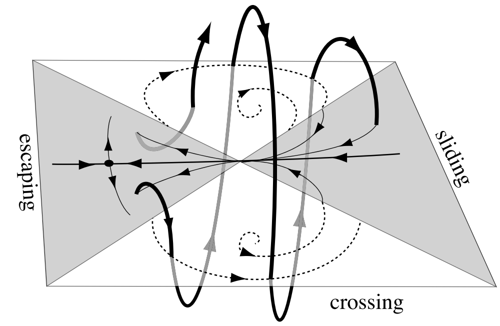

Fig. 27. The two-fold and non-deterministic chaos. For certain parameters the two-fold takes the form depicted, with a pseudosaddle in the escaping region and a family of canards passing through the two-fold. The folds are both invisible. The dotted curves are the paths followed by non-sliding orbit segments on successive crossings. A typical orbit is shown: orbits locally wrap around the singularity, and after finitely many crossings they slide, following a canard through the two-fold. The outward trajectory through the escaping region is not uniquely determined, yet wherever the orbit emerges, it again begins winding around to the sliding region and hence to the two-fold, to be ejected again in an undetermined direction. The resulting dynamics is chaotic but non-deterministic.

occur generically in control systems [35], they have not yet been identified in specific applications. These novel types of behaviour are deserving of further study in three dimensions and beyond.

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